كلية التربية الأساسية / حديثة | المستوى الثاني | فرع الفيزياء | فرع الكيمياء | 2021-2020

التفاضل و التكامل

CALCULUS

مساحة الدائرة

$$A = \pi r^2$$

قانون الدستور

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

متسلسلة تايلور

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \qquad -\infty < x < \infty$$

م . صالح محمد حسين

كلية التربية الاساسية – حديثة

قسم العلوم العامة

م. صالح محمد حسين التفاضل والتكامل المستوى 2 كلية التربية الأساسية / حديثة
Collage of Basic Education/Haditha Calculus - Level 2nd
(2020-2021)

مصطلحات ورموز

رسم الرمز	اسم الرمز	رسم الرمز	اسم الرمز
α	Alpha	w.r.t	With respect to
β	Beta	no.	Number
θ	Theta	Ex.	Example
δ or Δ	Delta	Sol.	Solution
ε	Epsilon	Def.	Definition
γ or Γ	Gamma	i.e	Which mean
θ	Theta	∴	So that
λ	Lamda	∵	Since
ζ	Zeta	→	Approach
η	Eata	⟹	Implies
μ	Mu	≡	Identical
φ or Φ	Phi	=	Equal
ψ	Psi	-ve	Negative
π	Pi	+ve	Positive
σ	Sigma	∃	There exist
τ	Tau	∀	For all
ρ	Rho		
∞	Infinity	∧	And
Iff	If and only if	∨	Or
fun.	function	∈	belongs to

Real Numbers: R (الأعداد الحقيقية)

R$=\{x: -\infty < x < \infty\}$

- If **a** and **b** are two real no.s then one of the following is true.
  some properties of R
  1- If $a > b$ then $-a < -b$
  2- If $a > b$ then $\frac{1}{a} < \frac{1}{b}$
  3- If $a < b$, $b < c$ then $a < c$
  4- If $a < b$ then $a + c < b + c$ $\forall$ real no. c
  5- If $a < b$, $c < d$ then $a + c < b + d$
  6- If $a < b$, c any + ve real no. then $a.c < b.c$
  7- If $a < b$, c any − ve real no. then $a.c > b.c$
  8- If $0 < a < b$, $0 < c < d$ then $a.c < b.d$
  9- If $a > b > 0$, $c > d > 0$ then $a.c > b.d$

Intervals (الفترات)

Def. :- An interval is a set of real no.s x having one of the following forms:-

1- Open interval (a, b) (الفترة المفتوحة)

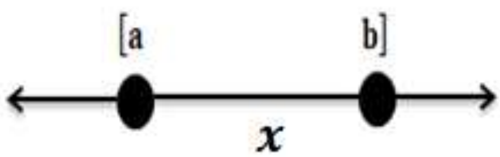
All real no.s x $\ni$ $a < x < b$

$(a, b) = \{x: a < x < b\}$

2- Closed interval [a, b] [الفترة المغلقة]

All real no.s x $\ni$ $a \le x \le b$

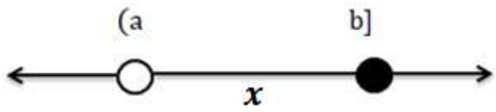
$[a, b] = \{x: a \le x \le b\}$



3- Half-open from the left or Half closed from the right.
$(a, b]$ (الفترة نصف مفتوحة من اليسار أو نصف مفتوحة من اليمين]

All real no.s $\boldsymbol{x} \ni a < x \leq b$

$(a, b] = \{x: \ a < x \leq b\}$



4- Half-open from the right or Half closed from the left.
$[a, b)$ [الفترة نصف مفتوحة من اليسار أو نصف مفتوحة من اليمين)

All real no.s $\boldsymbol{x} \ni a \leq x < b$

$[a, b) = \{x: \ a \leq x < b\}$

Notes:- **(ملحوظات)**

❖ $(a, \infty) = \{x: \ x > a, x \in R\}$

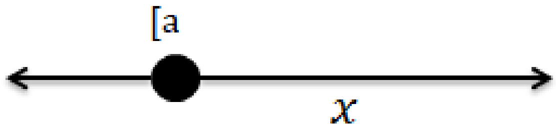
$\equiv a < x < \infty \equiv x > a$

❖ $(-\infty, a) = \{x: \ x < a, x \in R\}$

$\equiv -\infty < x < a \equiv x < a$

❖ $[a, -\infty) = \{x: \ x \geq a, x \in R\}$

$\equiv a \leq x < \infty \equiv x \geq a$



❖ $(-\infty, a] = \{x: \ x \leq a, x \in R\}$

$\equiv a \geq x > -\infty \equiv x \leq a$

Inequalities (متباينات)

$2x - 3 > 0, \quad x^2 - 5x - 24 \leq 0$

Examples:- Find the set of the following Inequalities.

1) $\mathbf{2 + 3x < 5x + 8}$

$2 - 8 < 5x - 3x$

$-6 < 2x$

$x > -3$

Sol. Set = $\{x: \ x > -3\} \equiv x > -3 \equiv (-3, \infty)$

2) $\mathbf{4 < 3x - 2 \leq 10}$

$4 + 2 < 3x - 2 + 2 \leq 10 + 2$

$6 < 3x \leq 12$

$2 < x \leq 4$

Sol. Set = $\{x: \ 2 < x \leq 4\} \equiv (2, 4]$

3) $\mathbf{\frac{7}{x} > 2 \quad , \quad x \neq 0 \quad , \quad x \in R}$

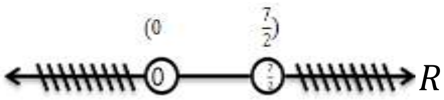
1st case:- $\mathbf{x > 0}$

$7 > 2x \Leftrightarrow \frac{7}{2} > x \Leftrightarrow x < \frac{7}{2}$

Sol. set 1st case = $(0\,,\frac{7}{2})$ R

2nd case:- $\boldsymbol{x < 0}$

$7 < 2x \;\Leftrightarrow\; \frac{7}{2} < x \;\Leftrightarrow\; x > \frac{7}{2}$



Sol. set 2nd case = $\boldsymbol{\phi}$

Sol. Set $=\left(\mathbf{0}\,,\frac{\mathbf{7}}{\mathbf{2}}\right) \cup \boldsymbol{\phi} = (\mathbf{0}\,,\frac{\mathbf{7}}{\mathbf{2}})$

4) $\boldsymbol{\frac{x}{x-3} < 4 \quad , \quad x \neq 3}$

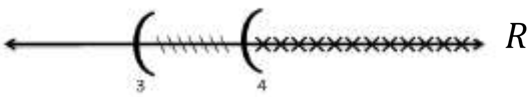
1st case:- $\boldsymbol{x > 3} \;\Leftrightarrow x - 3 > 0$

Multiply by $(x-3)$

$x < 4(x-3) \;\Leftrightarrow x < 4x - 12 \Leftrightarrow\; x - 4x < -12$

$\Leftrightarrow -3x < -12 \Leftrightarrow -\frac{1}{3}(-3x < -12) \Leftrightarrow x > 4$

Sol. set 1st case = $(4\,,\infty)$

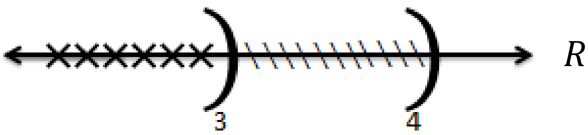


2nd case:- $\boldsymbol{x < 3}$

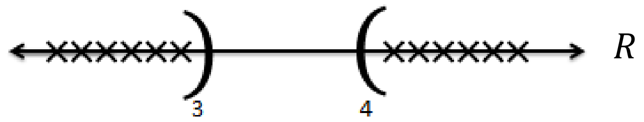
$x > 4(x-3) \;\Leftrightarrow x > 4x - 12 \Leftrightarrow\; x - 4x > -12$

$\Leftrightarrow -3x > -12 \Leftrightarrow -\frac{1}{3}(-3x > -12) \Leftrightarrow x < 4$

Sol. set 2nd case = $(-\infty\,,3)$



Sol. Set $=(4,\infty)\cup(-\infty,3) = R/[3,4]$

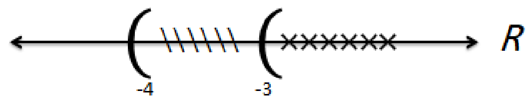


5) $(\boldsymbol{x}+\mathbf{3})(\boldsymbol{x}+\mathbf{4})>\mathbf{0}$

1st case:- $(x+3)>0 \;\wedge\; (x+4)>0$

$\Leftrightarrow\; x>-3 \;\wedge\; x>-4$

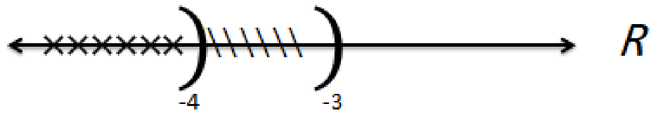
Sol. set 1st case $=(-3,\infty)$



2nd case :- $(x+3)<0 \;\wedge\; (x+4)<0$

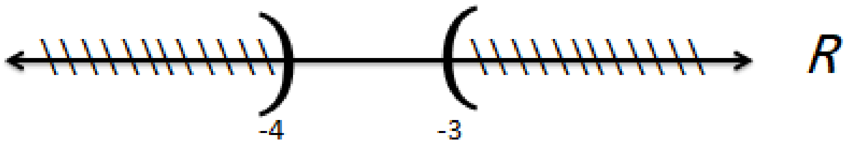
$\Leftrightarrow\; x<-3 \;\wedge\; x<-4$

Sol. set 2nd case $=(-\infty,-4)$



Sol. Set for both of the cases

$=(-3,\infty)\cup(-\infty,-4) = R/[-4,-3]$



6) $\frac{\boldsymbol{x}-\mathbf{1}}{\boldsymbol{x}^2+\boldsymbol{x}-\mathbf{6}}<\mathbf{0}$

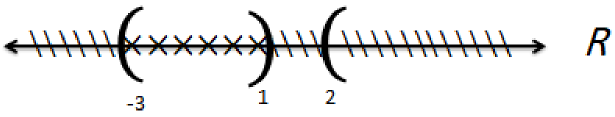
$x^2+x-6\neq 0 \;\Leftrightarrow\; (x+3)(x-2)\neq 0$

1st case:- $(x+3)(x-2) > 0$

$x-1<0 \Leftrightarrow x<1$

i. $(x+3)>0 \;\wedge\; (x-2)>0$

$x>-3 \;\wedge\; x>2$

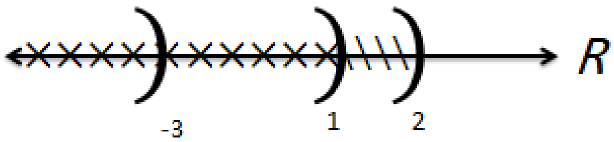


Sol. set i.1st case = ϕ

ii. $(x+3)<0 \;\wedge\; (x-2)<0$

$x-1<0 \Leftrightarrow x<1$

Sol. set ii.1st case $=(-\infty\,,-3)$



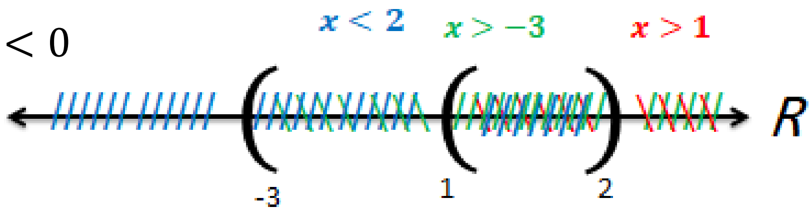
Sol. set 1st case = $\phi \cup (-\infty\,,-3) = (-\infty\,,-3)$

2nd case:- $(x+3)(x-2)<0$

$x-1>0 \Leftrightarrow x>1$

I. $(x+3)>0 \;\wedge\; (x-2)<0$

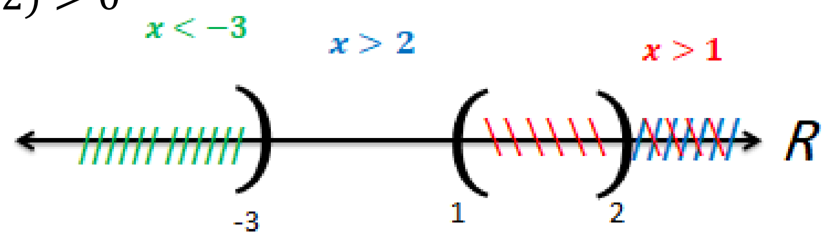
$x>-3 \;\wedge\; x<2$



Sol. set I. **2nd** case $=(1,2)$

II. $(x+3)<0 \;\wedge\; (x-2)>0$

$x<-3 \;\wedge\; x>2$



Sol. set II. **2nd** case $=\phi$

Sol. set **2nd** case = $=(1\,,2) \cup \phi = (1\,,2)$

Sol. Set of all cases $=(-\infty\,,-3) \cup (1,2)$

Absolute Value (القيمة المطلقة)

Def. :- The absolute value of a real no. $\boldsymbol{x}$ denoted by $|\boldsymbol{x}|$ is defined as follows:-

$$|\boldsymbol{x}| = \begin{cases} x & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$$

Ex.

$|0| = 0$, $|5| = 5$, $|-3| = 3$, $|1.8| = 1.8$,

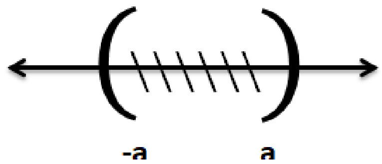
$|8-5| = 3$, $|4+2| = 6$

Note:-

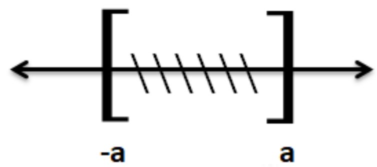
$$|\boldsymbol{x} - \boldsymbol{a}| = \begin{cases} x - a & \text{if } x \geq a \\ a - x & \text{if } x < a \end{cases}$$

Properties of Absolute Value (خواص القيمة المطلقة)

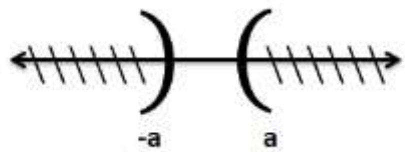
1- $|x| < a$ iff $-a < x < a$

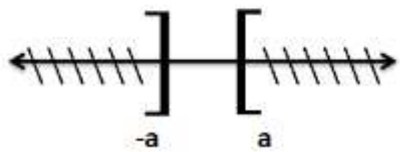


2- $|x| \leq a$ iff $-a \leq x \leq a$



3- $|x| > a$ iff $x > a$ or $x < -a$





4- $|x| \geq a$ iff $x \geq a$ or $x \leq -a$

Note:-

$|a| = |b|$

1- $a = b$

2- $a = -b$

3- $-a = b \iff a = -b \implies 2$

4- $-a = -b \iff a = b \implies 1$

Ex. Find sol. Set of the following:-

1- $|x - 4| < 5$

$-5 < x - 4 < 5$ (properties no. 1)

$\mathbf{-5 + 4 < x - 4 + 4 < 5 + 4}$

$-1 < x < 9$

Sol. Set = $\{x: -1 < x < 9\}$ = (1 , 9)

2- $|2x - 5| = 7$

This satisfies in two cases either

$2x - 5 = 7$ **or** $2x - 5 = -7$

$2x = 12$ $\qquad$ $2x = -2$

$x = 6$ $\qquad$ $x = -1$

Sol. Set = {-1 , 6}

3- $|x - 2| = |3x + 4|$

$x - 2 = 3x + 4$ **or** $x - 2 = -(3x + 4)$ **(Nots 1&2)**

$x - 3x = 2 + 4$ $\qquad$ $x - 2 = -3x - 4$

$-2x = 6$ $\qquad$ $4x = 2 - 4$

$x = -3$ $\qquad$ $x = \frac{-1}{2}$

Sol. Set = {-3 , $\frac{-1}{2}$}

Exercises (تمارين)

1- $-3 < \frac{3x-4}{5} \leq 2$

2- $\frac{2}{3} < \frac{10}{x}$

3- $|2x - 5| > 4$

4- $|6x| = |4x|$

5- $\left|\frac{2x-8}{2x-3}\right| \leq 4$

Relationship Between Square Root and Absolute Value (العلاقة بين الجذر التربيعي و القيمة المطلقة)

Recall from algebra that a number is called a ***square root*** of a if its square is. Recall also that every positive real number has two square roots, one positive and one negative; the positive square root is denoted by $\sqrt{a}$ and the negative square root by $-\sqrt{a}$. For example, the positive square root of 9 is $\sqrt{9}=3$, and the negative square root of 9 is $-\sqrt{9}=-3$.

Students who may have been taught to write $\sqrt{9}$ as ± 3 should stop doing so, since it is incorrect.

It is a common error to replace $\sqrt{a^2}$ by a .Although this is correct when a is nonnegative, it is false for negative a. For example, if $a=-4$, then

$$\sqrt{a^2}=\sqrt{(-4)^2}=\sqrt{16}=4\neq a$$

A result that is correct for all a is given in the following theorem

Theorem.

For any real number a, $\sqrt{a^2}=|a|$

Theorem:-

Let a , b two no.s then

1- $|a|=|-a|$
2- $|a.b|=|a||b|$
3- $\left|\frac{a}{b}\right|=\frac{|a|}{|b|}$ $\quad b\neq 0$

Functions (الدوال)

Note:-

From the definition of the Intervals Note that $a < x$

- Variable:- symbol x is represented to any no. from a set of numbers is called a variable.
- Constat:- a is represented only one number is called constant.

The Cartesian product (الضرب الديكارتي)

Def:- the **Cartesian product** of two sets *A* and *B*, denoted by $A \times B$, is the set of all ordered pairs (*a*, *b*) where *a* is in *A* and *b* is in *B*. in terms of set-builder notation, that is

$A \times B = \{(a, b): a \in A, b \in B\}$

This definition implies that

$$A \times B = \emptyset \quad \text{iff} \quad A = \emptyset \quad \text{or} \quad B = \emptyset$$

The sets $A \times B$ and $B \times A$ are not identical $A \times B \neq B \times A$

$A \times B = B \times A \quad \text{iff} \quad A = B \quad \text{and} \quad A = \emptyset \quad \text{or} \quad B = \emptyset$

Functions and its Algebra (الدوال وجبرها)

Def:- A function of from a set D to a set R is a rule that assigns a single element $y \in R$ to each element $x \in D$

$$f = \{(x, f(x)/x \in D\}$$

- نقول عن المتغير y أنه دالة لمتغير أخر x إذا أعطية قاعدة أو وسيلة بحيث أن تقابل كل قيمة لـ x في مداها قيمة لـ y.

i.e

Let $f: D \to R$ is function iff for each $\in D$, there is one and only one $y \in R$ satisfying $f(x) = y$.

If (x, y_1) and (x, y_2) are elements in F then $y_1 = y_2$

1- The Domain (منطلق الدالة) مجال الدالة

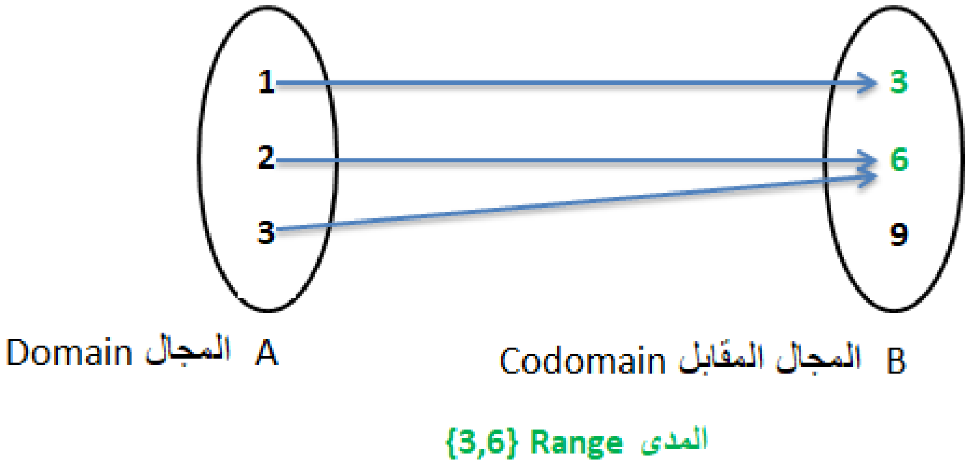
The set D is called the domain of f is the set of all x……. Components occurring in the ordered pairs of f.

Denoted by dom(*f*) =$D_f = \{x$: F is defined$\}$

$$= \{x \in D: (x, y) \in f \text{ for some } y \in R\}$$

2- The Codomain المجال المقابل

The codomain is for all element in too set R and denoted by Cod_F such that $Cod_F = \{y: y \in R\}$

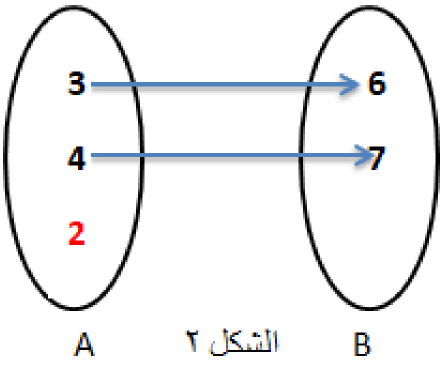


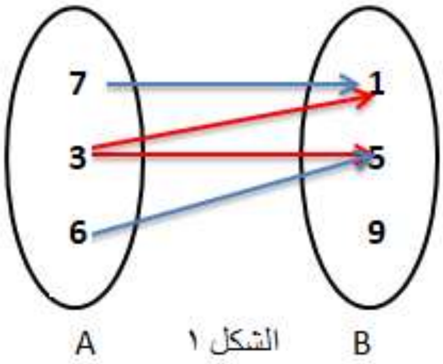
3- The Range (المدى)

Is the set of all second components of element of f denoted by.

$$\text{Rang}_f = R_f = \{f(x) : x \in \text{Dom}(f)\}$$

$$= \{y : (x, y) \in F \ \text{ for som } x \in D\ \}$$

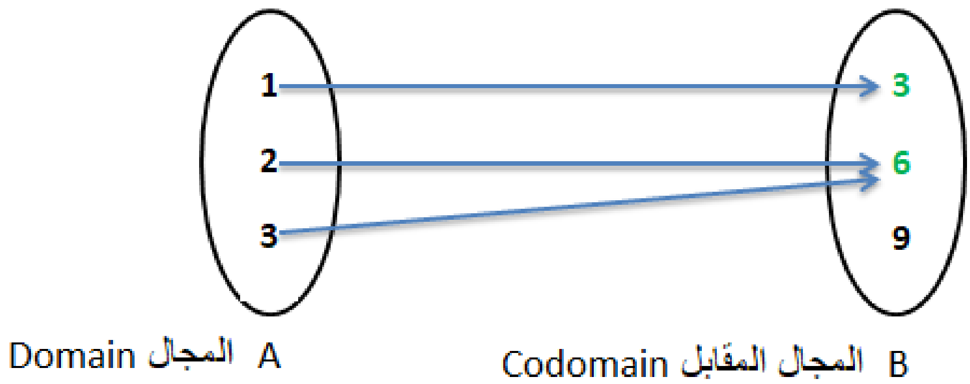




في الشكلين اعلاه (2&1) لا تمثل دوال لان في الشكل 1 هناك العنصر 3 له علاقة في عنصرين من عناصر المجال المقابل وهذا يعارض شرط ان لكل عنصر من عناصر المجال له علاقة بعنصر واحد فقط في المجال المقابل ، اما الشكل 2 فأن العنصر 2 ليس له علاقة مع اي عنصر في المجال المقابل.

How can write the function as ordered pairs?

Let try about the previous example.



$$\boldsymbol{f} : \mathbf{A} \rightarrow \mathbf{B}$$

$$f = \{(1, 3), (2, 6), (3, 6)\}$$

Ex. Determine which of the following sets is a function. If it is a function, what is its domain (D_f) and rang (R_F)?

a) $f = \{(1,4),(3,6),(-3,5),(0,0),(5,0)\}$

Sol.

$$D_f = \{1,3,-3,0,5\} \quad \& \quad R_f = \{4,6,5,0\}$$

b) $g = \{(1,3),(3,6),(3,5),(4,4),(5,0)\}$

Sol.

g is not a function $(\mathbf{3},6)$ and $(\mathbf{3},5)$ have the same x-coordinate.

- The variable x is called the independent variable of f (or argument) and the variable y is called the dependent variable of f.

Def:- Let f be fun. then the graph of fun. f is the set of all points (x, y) is the ordered pairs in

Ex.

Let f be the fun. $f(x) = \sqrt{5-x}$ find its domain and range?

$$D_f = \{x: 5-x \geq 0\} = \{x: x \leq 5\} = (-\infty, 5]$$

$$R_f = \{f(x): x \in D_f\} = \{f(x): x \leq 5\} = [0, \infty)$$

Ex. Let g be fun. defined as follows

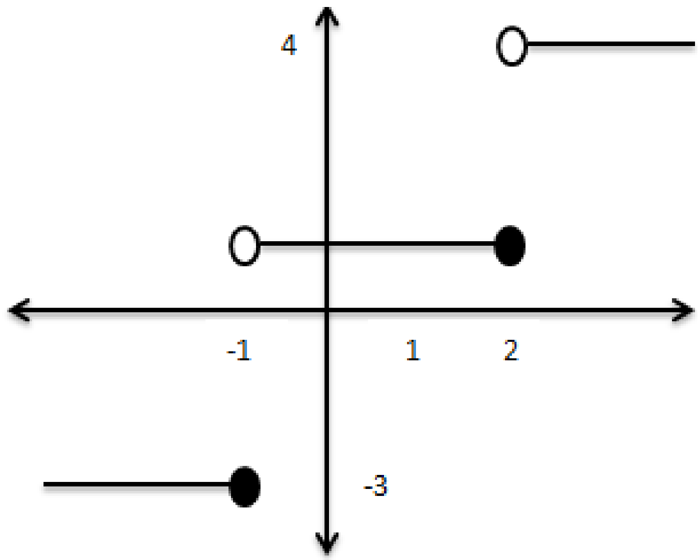
$$y = g(x) = \begin{cases} -3 & \text{if} \quad x \leq -1 \\ 1 & \text{if} \quad -1 < x \leq 2 \\ 4 & \text{if} \quad 2 < x \end{cases}$$

Find the domain, range, and sketch the fun.

Sol.

$D_g = R$

$R_g = \{-3,1,4\}$



Kinds of Functions (انواع الدوال)

Functions are divided into two types.

1- Algebraic function (الدالة الجبرية)

i. Polynomial function (دالة متعددة الحدود)

$$y = a_\circ + a_1x + a_2x^2 + \cdots \dots \dots \dots \dots + a_nx^n$$

Such that $a_\circ, a_1, a_2, \dots . a_n \in R$

Ex. 1. $y = x^3 + 2x + 7$

2. $y = x + 5$

3. $y = 2x$

ii. Regular function (دالة قياسية أو الكسرية) $y = \frac{\text{متعددة الحدود}}{\text{متعددة الحدود}}$

$$y = \frac{2x+1}{x^2+3x} \quad , \quad y = \frac{1}{x^2}$$

iii. Radical function (دالة جذرية)

1. $y = \frac{\sqrt{x^2-1}}{x+2}$ 2. $y = \sqrt{x^2+1}$

2- Non algebraic function (الدالة غير الجبرية)

i. Trigonometric function (دوال مثلثية)

1. $y = \sin x$ 2. $y = \cos x$ 3. $y = \tan x$

ii. Exponential function (دالة أسية)

1. $y = 2^x$ 2. $y = (\frac{1}{2})^x$ 3. $y = e^x$

iii. Logarithmic function (دالة لوغارتمية)

1. $y = log_{10}^{x}$ (اللوغاريتم العشري)

2. $y = lnx$ (اللوغاريتم الطبيعي)

iv. Absolute value function (دالة القمية المطلقة)

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Def:-

1- $f(x)$ is an **even** fun. if $f(-x) = f(x)$ for every x in the domain.

2- $f(x)$ is an **odd** fun. if $f(-x) = -f(x)$ for every x in the domain.

Exs.

1) $\sin\theta$

$\sin(-\theta) = -\sin(\theta)$ is odd fun.

2) $\cos\theta$

$\cos(\theta) = \cos(-\theta)$ is even fun.

3) $f(x) = x^2 \cos x$

$$f(-x) = (-x)^2 \cos(-x) = x^2 \cos x = f(x)$$

$\therefore f(x)$ is even fun.

4) $f(x) = \frac{x^2-1}{\sin x}$

$$f(-x) = \frac{(-x)^2 - 1}{\sin(-x)} = \frac{x^2 - 1}{-\sin x} = -\left(\frac{x^2 - 1}{\sin(x)}\right) = -f(x)$$

$\therefore f(x)$ is odd fun.

Sum, Differences, product and quotient of Functions (الجمع، الطرح، الضرب، و القسمة للدوال)

If $f(x)$ and $g(x)$ are two functions then :-

1- The sum $f + g$ is the fun.

$(f + g)(x) = f(x) + g(x)$

Domain $(f + g) = \text{domain}(f) \cap \text{domain}(g)$

2- The differences $f - g$ is the fun.

$(f - g)(x) = f(x) - g(x)$

Domain $(f - g) = \text{domain}(f) \cap \text{domain}(g)$

3- The product $f.g$ is the fun.

$(f.g)(x) = f(x).g(x)$

Domain $(f.g) = \text{domain}(f) \cap \text{domain}(g)$

4- The quotient $\frac{f}{g}$ is the fun.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$

Domain $\left(\frac{f}{g}\right) = \text{domain}(f) \cap \text{domain}(g) \backslash \{x: g(x) = 0\}$

Ex. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$

Find $(f + g)$, $(g - f)$, $(f - g)$, $\left(\frac{f}{g}\right)$, $\left(\frac{g}{f}\right)$, $(f.g)$ and its domain.

Sol.

$\text{domain}(f) = [0, \infty)$, $\text{domain}(g) = (-\infty, 1]$

$1 - x \geq 0$

$-x \geq -1$

$x \leq 1$

$\text{Domain}(f) \cap \text{Domain}(g) = [0,1]$

- $(f+g)(x)=\sqrt{x}+\sqrt{1-x}$ Domain$(f+g)=[0,1]$
- $(f-g)(x)=\sqrt{x}-\sqrt{1-x}$ Domain$(f-g)=[0,1]$
- $(g-f)(x)=\sqrt{1-x}-\sqrt{x}$ Domain$(g-f)=[0,1]$
- $(f.g)(x)=f(x).g(x)=\sqrt{x}\sqrt{1-x}=\sqrt{x-x^2}$

$$D_{f.g}=[0,1]$$

- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{\sqrt{x}}{\sqrt{1-x}}=\sqrt{\frac{x}{1-x}}$ $D_{\frac{f}{g}}=[0,1)$
- $\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}=\frac{\sqrt{1-x}}{\sqrt{x}}=\sqrt{\frac{1-x}{x}}=\sqrt{\frac{1}{x}-1}$ $D_{\frac{g}{f}}=(0,1]$

Composition of Functions (تركيب الدوال)

Composition is another method for combining functions. In this operation the output from one function becomes the input to a second function.

Def. :- If ***f*** and ***g*** are functions, the **composite** function ***f* ∘ *g*** ("***f*** composed with ***g***") is defined by:

$$(f\circ g)(x)=f(g(x))$$

The domain of ***f* ∘ *g*** consists of the numbers x in the domain of ***g*** for which **g(*x*)** lies in the domain of ***f***.

Figure (1)

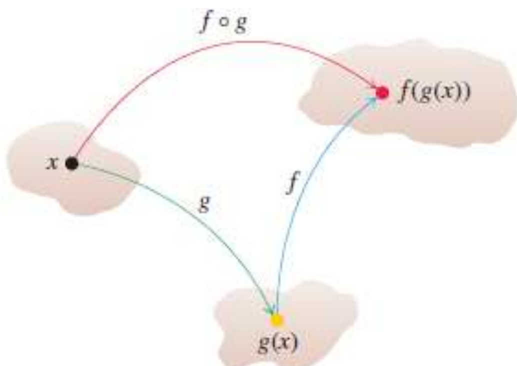


Figure (2)

The definition implies that $\boldsymbol{f \circ g}$ can be formed when the range of $\boldsymbol{g}$ lies in the domain of $\boldsymbol{f}$. To find $\boldsymbol{(f \circ g)(x)}$, *first* find $\boldsymbol{g(x)}$ and *second* find $\boldsymbol{f(g(x))}$. Figure (1) pictures $f \circ g$ as a machine diagram, and Figure (2) shows the composition as an arrow diagram.

To evaluate the composite function $g \circ f$ (when defined), we find $f(x)$ first and then find $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

Ex. If $f(x) = \sqrt{x}$ and $g(x) = x + 1$

Find **(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Sol.	**Composition**	**Domain**
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but $g(x)$ belongs to the domain of f only if $x + 1 \geq 0$,that is to say, when $x \geq -1$.

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then

$(f \circ g)(x) = (\sqrt{x})^2 = x$. However,the domain of $f \circ g$ is $[0, \infty)$, not (∞, ∞), since $\sqrt{x}$ requires $x \geq 0$.

Ex. Let $f(x) = 2x + 1$ find the fun. $g(x)$ in which

$(f \circ g)(x) = x^3$.

Sol.

$(f \circ g)(x) = f(g(x)) = 2g(x) + 1 = x^3 \Leftrightarrow 2g(x) = x^3 - 1$

$$g(x) = \frac{x^3 - 1}{2}$$

Exercises (تمارين)

1) Let $f(x) = -3x + 2$ find the fun. $g(x)$ in which

$(g \circ f)(x) = x$.

2) Find the domain and rang of the following fun.s

   a) $y = x^2 + 1$
   b) $y = \sqrt{6 - x}$
   c) $y = \frac{1}{\sqrt{x}}$
   d) $y = \sqrt{8 - \sqrt{x}}$
   e) $y = \frac{x-1}{x}$
   f) $y = \frac{1-|x|}{|x|}$

3) Find $f + g$, $f - g$, $f.g$, $\frac{f}{g}$, $\frac{g}{f}$ and its domain

a) $f(x) = \frac{x}{2}$, $g(x) = \sqrt{x + 1}$

b) $f(x) = \frac{1}{x-2}$, $g(x) = \frac{1}{\sqrt{x-1}}$

4) Find $f \circ g$ and $g \circ f$ and its domain of

   a) $f(x) = \sqrt{2 - x}$ $g(x) = \sqrt{x - 2}$
   b) $f(x) = \sqrt{3 + x^2}$ $g(x) = \frac{1}{x}$

Limit (الغاية)

We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

نستخدم الغايات لوصف الطريقة التي تختلف بها الدالة. تختلف بعض الدوال باستمرار ؛ التغييرات الصغيرة في x تنتج فقط تغييرات صغيرة في $f(x)$. ودوال أخرى يمكن أن يكون القيم التي تقفز أو تتغير بشكل متقطع أو تميل إلى الزيادة أو النقصان بدون حدود. مفهوم الغاية تعطي طريقة دقيقة للتمييز بين تلك السلوكيات.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance d of c (Fig. 3).

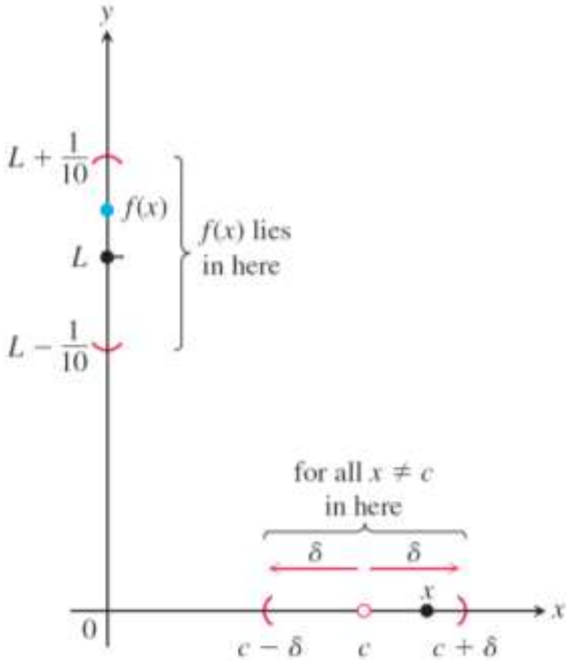


Fig.(3) How should we define $\delta > 0$ so that keeping x within the interval $(c-\delta, c+\delta)$ will keep $f(x)$ within the interval (L-1/10),(L+1/10)?

But that in itself is not enough, because as x continues on its course toward c, what is to prevent $f(x)$ from jumping around within the interval from L - (1/10) to L + (1/10) without tending toward L? We can be told that the error can be no more than 1/100 or 1/1000 or 1/100,000. Each time, we find a new δ -interval about c so that keeping x within that interval satisfies the new error tolerance.

And each time the possibility exists that $f(x)$ might jump away from L at some later stage.
We can present a matching distance d that keeps x "close enough" to c to keep $f(x)$ within that ε-tolerance of L (Fig.4). This leads us to the precise definition of a limit.

Def: - Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L$$

if, for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ a such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

To visualize the definition, imagine machining a cylindrical shaft to a close tolerance .The diameter of the shaft is determined by turning a dial to a setting measured by a variable x. We try for diameter L, but since nothing is perfect we must be satisfied with a diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The number δ is our control tolerance for the dial; it tells us how close our dial setting must be to the setting $x = c$ in order to guarantee that the diameter $f(x)$ of the shaft will be accurate to within ε of L.

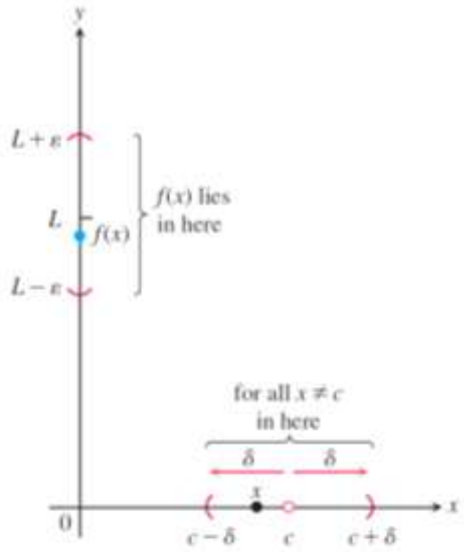


Fig.(4) The relation of δ and ε in the definition of limit.

As the tolerance for error becomes stricter, we may have to adjust δ. The value of δ, how tight our control setting must be, depends on the value of ε, the error tolerance.The definition of limit extends to functions on more general domains. It is only required that each open interval around c contains points in the domain of the function other than c.

Theorem:-

If L, M, c, and k are real no.s and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c} (f(x) + g(x)) = L + M$
2. *Difference Rule*: $\lim_{x\to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule*: $\lim_{x\to c} (k.f(x)) = k.L$
4. *Product Rule*: $\lim_{x\to c} (f(x).g(x)) = L.M$
5. *Quotient Rule*: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule*: $\lim_{x\to c} [f(x)]^n = L^n, n$ a positive integer
7. *Root Rule*: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, n$ a positive integer

(if n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Ex. If $f(x) = 2x + 5$, Find: $\lim_{x\to 1} f(x)$.

Sol.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(2x+5) = \lim_{x\to 1} 2x + \lim_{x\to 1} 5 = 2\lim_{x\to 1} x + 5$$
$$= 2(1) + 5 = 7$$

Ex. If $f(x) = \frac{x^2-3x+2}{x-2}$, Find : $\lim_{x\to 2} f(x)$.

Sol.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \frac{4-6+2}{2-2} = \frac{0}{0}$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)} = \lim_{x\to 2}(x-1)$$
$$= 2 - 1 = 1$$

Limits of Polynomials (الغاية لمتعددات الحدود)

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots \ldots \ldots \ldots + a_\circ$ is any polynomial fun. ,Then

$$\lim_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots \ldots \ldots \ldots + a_\circ$$

Limits of Quotients of Polynomials

If $f(x)$ and $g(x)$ are polynomials fun. , Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \quad , \quad g(c) \neq 0$$

Ex. If $f(x) = (x^2 + 3x - 1)$, Find : $\lim_{x\to -1} f(x)$.

Sol.

$$\lim_{x\to -1} f(x) = (-1)^2 + 3(-1) - 1 = -3$$

Ex. Find $\lim_{x\to 2}\frac{x^2+2x+4}{x+2}$

Sol.

$$\lim_{x\to 2}\frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{x + 2} = \frac{12}{4} = 3$$

Ex. Find $\lim_{x\to 1}\frac{x^3-1}{x-1}$, $x \neq 1$

Sol.

$$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{x^3-1}{x-1} \qquad \boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

$$= \lim_{x\to 1}\frac{(x - 1)(x^2 + x + 1)}{(x - 1)} = \lim_{x\to 1}(x^2 + x + 1) = 3$$

Ex. Find $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$, $h\neq 0$

Sol.

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$ (الضرب بمرافق البسط)

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\left(\sqrt{x+h}\ +\ \sqrt{x}\right)}=\frac{1}{\sqrt{x}\ +\ \sqrt{x}}=\frac{1}{2\sqrt{x}}$$

One-Sided Limits (الغايات من جهة واحدة)

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only. These allow us to describe functions that have different limits at a point,depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**. If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. From the left, it is a **left-hand limit** or **limit from the left.**

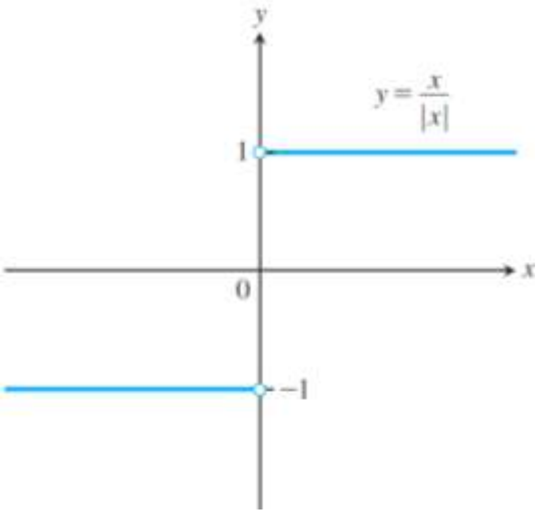


Fig.(5) Different right-hand and left-hand limits at the origin.

The function $f(x) = \frac{x}{|x|}$ (Fig. 5) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we only consider the values of $f(x)$ on an interval (c, b), where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. In this case we write

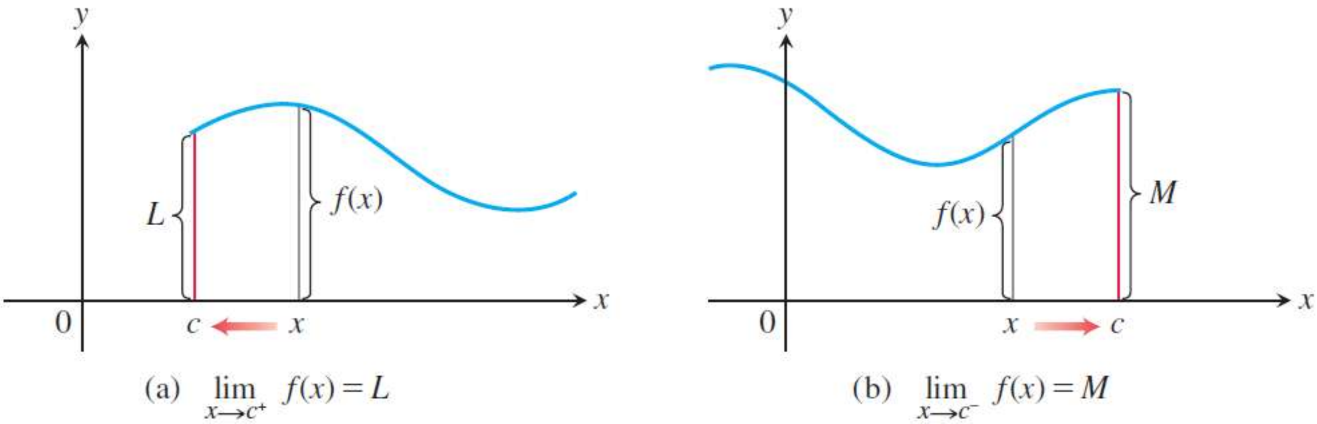
$$\lim_{x \to c^+} f(x) = L$$

The notation "$x \to c^+$" means that we consider only values of $f(x)$ for x greater than c. We don't consider values of $f(x)$ for $x \leq c$ Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M$$

The symbol "$x \to c^-$ " means that we consider the values of f only at x-values less than c. These informal definitions of one-sided limits are illustrated in (Fig. 6). For the function $f(x) = \frac{x}{|x|}$ in (Fig.5) we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$



Fig(6) (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c

Ex. $(x) = \sqrt{x}$, Find $\lim_{x \to 0} f(x)$.

Sol.

Dom. of $f(x)$ is $x \geq 0$

Since $\sqrt{x}$ is not defined for –ve values x so

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$$

Ex. $(x) = |x|$, Find $\lim_{x \to 0^+} |x|$.

Sol.

$$\text{Since} |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Then $f(x) = x$ where $x \to 0^+$

So $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

Ex. $(x) = \sqrt{1-x}$, Find $\lim_{x \to 1} f(x)$.

Sol.

Dom. $1 - x \geq 0 \quad \Leftrightarrow x \leq 1$

Since $\sqrt{1-x}$ is not defined for $x > 1$, so

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1-x} = \sqrt{1-1} = 0 = \lim_{x \to 1} f(x)$$

Theorem:

1) $\lim_{x\to 0} \sin x = 0$
2) $\lim_{x\to 0} \cos x = 1$
3) $\lim_{x\to 0} \frac{\sin x}{x} = 1$
4) $\lim_{x\to 0} \frac{\cos x}{x} = \infty$
5) $\lim_{x\to 0} \frac{1}{x} = \infty$

Ex. Find the following limits:-

a) $\lim_{x\to 0} \frac{\sin 3x}{x} = 3\lim_{x\to 0} \frac{\sin 3x}{3x}$

$\because$ as $x \to 0 \quad \Longrightarrow \quad 3x \to 0$

$$= 3 \lim_{3x\to 0} \frac{\sin 3x}{3x} = 3 \times 1 = 3$$

b) $\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x\to 0} \frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} = \frac{5}{3} \lim_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 3x}{3x}} = \frac{5}{3} \frac{\lim_{5x\to 0} \frac{\sin 5x}{5x}}{\lim_{3x\to 0} \frac{\sin 3x}{3x}} = \frac{5}{3}$

Limits At Infinity (الغاية عند اللانهاية)

Def:-

Limits as $x \to \infty$ or $x \to -\infty$

1. The limit of the fun. $f(x)$ as x approaches infinity is the no. L.

$\lim_{x\to\infty} f(x) = L$ if

Given any $\varepsilon > 0$ there exists a no. $M > 0$ such that for all x,

$$M < x \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

2. The limit of the fun. $f(x)$ as x approaches negative infinity is the no. L.
   $\lim_{x\to-\infty} f(x) = L$ if

Given any $\varepsilon > 0$ there exists a no. $N < 0$ such that for all x,

$$x < N \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

Theorem:

If $f(x) = k$ (for any no. k)

1. $\lim_{x\to\infty} k = \lim_{x\to-\infty} k = k$
2. If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$

When L and M (are positive real no.s) then:-

a) $\lim_{x\to\infty}\big(f(x) \mp g(x)\big) = \lim_{x\to\infty} f(x) \mp \lim_{x\to\infty} g(x) = L \mp M$

b) $\lim_{x\to\infty}\big(f(x).g(x)\big) = LM$

c) $\lim_{x\to\infty} kf(x) = k\lim_{x\to\infty} f(x) = kL$ (for any no. k)

d) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

These results hold for $x \to -\infty$

Ex. Find the limit of :-

1. $\lim_{x\to\infty} \frac{x}{7x+4}$

عندما نعوض ونجد أن النتيجة $\frac{\infty}{\infty}$ نقسم على اكبر أس

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{7x+4}{x}} = \lim_{x\to\infty} \frac{1}{7+\frac{4}{x}} = \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} 7 + \lim_{x\to\infty} \frac{4}{x}} = \frac{1}{7}$$

2. $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1}{x} = 0 \cdot 0 = 0$

Theorem:

1. $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
2. $\lim_{x\to\infty} \frac{\cos x}{x} = 0$
3. $\lim_{x\to\infty} \frac{1}{x} = 0$

Ex. Find

1. $\lim_{x\to\infty} (2 + \frac{\sin x}{x})$

$= \lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{\sin x}{x} = 2 + 0 = 2$

2. $\lim_{x\to\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\infty} \frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}} = \frac{2-0+0}{3+0} = \frac{2}{3}$

Infinite Limits (الغايات اللانهاية)

Def:-

1. The limit of the fun. $f(x)$ as $x \to a$ is the infinity such that $a \notin D_f$

$\lim_{x\to a} f(x) = \infty$

if $\forall\, M > 0$ There exists $\delta > 0$ such that

$\forall x \in D_f\,, \quad 0 < |x-a| < \delta \;\Rightarrow f(x) > M$

2. The limit of the fun. $f(x)$ as $x \to a$ is the negative infinity such that $a \notin D_f$

$\lim_{x\to a} f(x) = -\infty$

if $\forall\, M < 0$ There exists $\delta > 0$ such that

$\forall x \in D_f\,, \quad 0 < |x-a| < \delta \;\Rightarrow f(x) < M$

Ex.

1. $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$
2. $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$
3. $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$
4. $\lim\limits_{x\to 2^+} \frac{1}{x^2-4} = \frac{1}{0} = \infty$
5. $\lim\limits_{x\to 2^-} \frac{1}{x^2-4} = \frac{1}{0} = -\infty$
6. $\lim\limits_{x\to \infty} \frac{2x^3+2x-1}{x^2-5x+2} = \lim\limits_{x\to \infty} \frac{2+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}+\frac{2}{x^3}} = \frac{2+0-0}{0+0+0} = \infty$

Continuity (الأستمرارية)

Def:-

The fun. $y = f(x)$ is continuous at $x = c$ iff all three of the following statements are true:-

1. $f(c)$ is exists (c is in the domain of f).
2. $\lim\limits_{x\to c} f(x)$ is exists (f has a limit as $x \to c$).
3. $\lim\limits_{x\to c} f(x) = f(c)$ (the limit equals the fun. value).

Continuous function

Def:-

A fun. is continuous if it is continuous at each point of its domain.

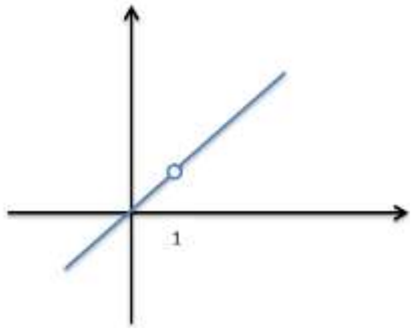
Discontinuity at a point

Def:-

If a fun. f is not cont. at a point c, we say that f is discontinuous at c and call c a point of discontinuity of f .

Examples:-

1. $f(x)=\frac{2x^2-3x+1}{x-1}$ is f cont. at $x=1$?



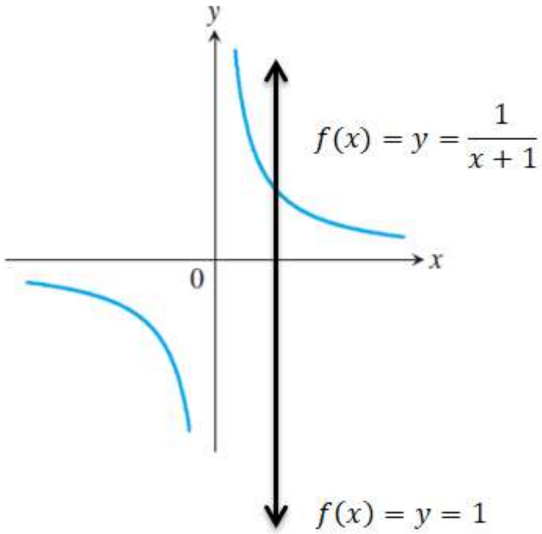
$D_f = R \setminus \{1\}$

$\therefore f(1)$ not exist

$\therefore f$ is discount. At $x=1$

$$f(x)=\frac{2x^2-3x+1}{x-1}=\frac{(2x-1)(x-1)}{(x-1)}=2x-1$$

2. $f(x)=\begin{cases}\frac{1}{x+1} & x\neq -1\\ 1 & x=-1\end{cases}$ is f cont. at $x=-1$



i) $f(-1)=1$ exist

ii) $\lim_{x\to -1} f(x)=\lim_{x\to -1}\frac{1}{x+1}$ not exist

$\therefore f(x)$ is not cont. at $x=-1$

3. $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is f cont. at $x = 0$

Sol.

i) $f(0) = 1$ is exist

ii) $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ is exist as $x \to 0$

iii) $\lim\limits_{x \to 0} f(x) = f(0) = 1$

$\therefore$ f is cont.

4. $f(x) = \begin{cases} \frac{x^2+x-6}{x^2-4} & x \neq 2 \\ \frac{5}{4} & x = 2 \end{cases}$ is f cont. at $x = 2$

Sol.

i) $f(2) = \frac{5}{4}$

ii) $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \frac{x^2+x-6}{x^2-4} = \lim\limits_{x \to 2} \frac{(x+3)(x-2)}{(x-2)(x+2)} = \frac{5}{4}$

iii) $f(2) = \lim\limits_{x \to 2} f(x) = \frac{5}{4}$

$\therefore$ f is cont.

Note:

1. Evry polynomial function

   $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots \ldots \ldots \ldots + a_\circ$ is cont.
   $(\lim_{x \to c} f(x) = f(c))$

2. Evry quotient $\frac{f(x)}{g(x)}$ of polynomial function is cont. except where $g(x) = 0$

   $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$ at every point c at which g does not equal zero.

Ex.

$f(x) = \frac{1}{x}$ is cont. at each point except at zero because $f(0)$ is not defined.

Theorem:-

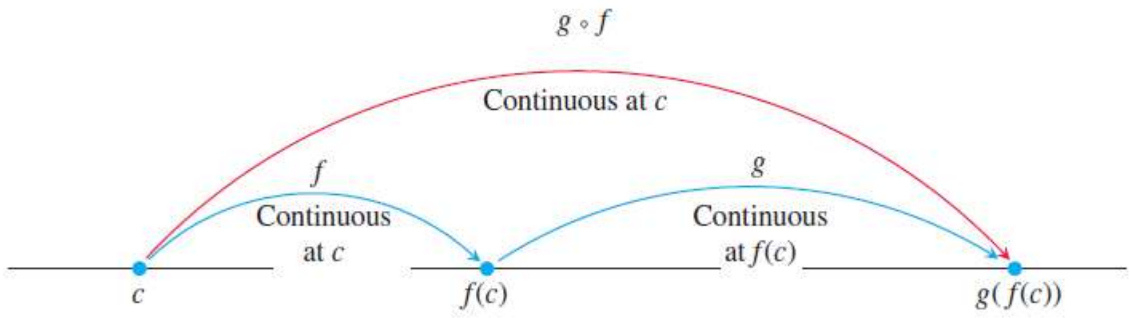
If the functions *f* and *g* are continuous at *x* = *c*, then the following algebraic combinations are continuous at *x* = *c*.

1. *Sums*: $f + g$
2. *Differences* : $f - g$
3. *Constant Multiples*: $k.f$ for any number *k*
4. *Products*: $f \cdot g$
5. *Quotients*: $\frac{f}{g}$, provided $g(c) \neq 0$
6. *Powers*: f^n, n a positive integer
7. *Roots*: $\sqrt[n]{f}$ provided it is deined on an interval containing *c*, where *n* is a positive integer

Theorem:-

If f is cont. at c and g is cont. at $f(c)$ then the composite $g \circ f$ is cont. at c.

$$\lim_{x \to c}(g \circ f)(x) = g(f(c))$$



Fig(7) Compositions of continuous functions are continuous

Ex.

Show that $y = \left|\frac{x \sin x}{x^2+2}\right|$ cont. at every value of x

Sol.

The fun. y is the composite of fun.s

$$f(x) = \frac{x \sin x}{x^2 + 2} \quad , \quad g(x) = |x|$$

$\therefore f$ is cont. and g is cont. then $y = g \circ f$ is also cont.

The derivative of a function (مشتقة الدالة)

The derivative of a fun. f is the fun. f' whose value at x is defined by the eq.

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ whenever the limit exists

A function is Differentiable at a point

(دالة قابلة للأشتقاق عند نقطة ما)

A fun. that has a derivative at appoint x is said to be differentiable at x . denoted by $\frac{dy}{dx}, \frac{df}{dx}, D_y,\ D_f$

Ex.

Find the derivative of $f(x) = x^2$

Sol.

$$f'(x) = \frac{dy}{dx} = D_f = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

HW. Find $\frac{dy}{dx}$ if $y = \sqrt{x}$ and $x > 0$

Some Rules of Derivatives

1. If $y = f(x) = c$, where c is constant, then $\frac{dy}{dx} = 0$
2. Power rule for positive integer (power of x) $y = x^n$ if n is positive integer then $\frac{d}{dx}(x^n) = nx^{n-1}$
3. The constant multiple rule.

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{df(x)}{dx} = cf'(x)$$

4. The sum rulr

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} = u' + v'$$

5. The product rule

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

6. Positive integer power of a diff. fun.
   If u is a diff. fun., n is power of u then
   $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
7. Negative integer power of a diff. fun.
   If u is a diff. fun., n is power of u then
   $\frac{d}{dx}u^{-n} = -nu^{-n-1}\frac{du}{dx}$

8. The quotient rule.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex.s

1) If $y = 5$ then $\frac{dy}{dx} = 0$

2) If $y = x$ then $\frac{dy}{dx} = 1x^{1-1} = 1$

3) If $y = -8x^4$ then $\frac{dy}{dx} = -8(4x^3) = -32x^3$

4) If $y = -2x^5 - 3x^2 + 6$ then $\frac{dy}{dx} = -10x^4 - 6x + 0$

$$= -10x^4 - 6x$$

5) If $y = (x^2 + 2)(x^3 + 3x + 1)$ then

$\frac{dy}{dx} = (x^2 + 2)(3x^2 + 3) + (x^3 + 3x + 1)(2x)$

6) If $y = \left(x^3 - \frac{x}{2}\right)^6$ then $\frac{dy}{dx} = 6\left(x^3 - \frac{x}{2}\right)^5 (3x^2 - \frac{1}{2})$

7) If $y = \frac{2x^3+3x-1}{x^2-1}$ then $\frac{dy}{dx} = \frac{(x^2-1)(6x^2+3)-(2x^3+3x-1)(2x)}{(x^2-1)^2}$

8) If $y = (2x^2 - 5x^{-2})^{-5}$ then

$\frac{dy}{dx} =$ -5$(2x^2 - 5x^{-2})^{-6}(4x + 10x^{-3})$

Implicit Differentiation (الأشتقاق الضمني)

Ex. If $y^2 = x$ Find $\frac{dy}{dx}$

$2y\frac{dy}{dx} = 1$ $\therefore \frac{dy}{dx} = \frac{1}{2y}$

Ex. Find the derivative of the implicit fun. $x^3 - xy + y^3 = 1$

Sol.

$$3x^2 - \left(x\frac{dy}{dx} + y.1\right) + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2 - x) = y - 3x^2$$

$$\frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$$

Ex.

If $x = y\sqrt{1-y^2}$ Find $\frac{dy}{dx}$?

Sol.

$$\frac{dx}{dy} = \frac{1}{2}y(1-y^2)^{-\frac{1}{2}}(-2y) + \sqrt{1-y^2}$$

$$= \frac{-y^2}{\sqrt{1-y^2}} + \sqrt{1-y^2} = \frac{-y^2+1-y^2}{\sqrt{1-y^2}} = \frac{1-2y^2}{\sqrt{1-y^2}}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1-2y^2}{\sqrt{1-y^2}}} = \frac{\sqrt{1-y^2}}{1-2y^2}$$

Derivatives of higher order

$y' = \frac{dy}{dx}$ First derivative

$y'' = \frac{dy'}{dx'} = \frac{d}{dx}(\frac{dy}{dx})$ Second derivative

$y^n = f^n(x) = \frac{dy^n}{dx^n}$

Ex. If $y = f(x) = x^4 - 3x^3 + 1$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 4x^3 - 9x^2 = y^-$ and $\frac{d^2y}{dx^2} = 12x^2 - 18x = y''$

Ex. If $y = 3x^4 - 5x^3 + 6x - 7$ find $\frac{d^4y}{dx^4}$

Sol.

$$\frac{dy}{dx} = 12x^3 - 15x^2 + 6$$

$$\frac{d^2y}{dx^2} = 36x^2 - 30x$$

$$\frac{d^3y}{dx^3} = 72x - 30$$

$$\frac{d^4y}{dx^4} = 72$$

Ex. Find y^n to $y = \frac{(x+1)}{(x-1)}$ if we use $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Sol.

$$y = \frac{(x-1)+2}{(x-1)} = 1 + \frac{2}{(x-1)} = 1 + 2(x-1)^{-1}$$

$$y' = -2(x-1)^{-2}$$

$$y'' = 4(x-1)^{-3}$$

$$y''' = -12(x-1)^{-4}$$

$$y'''' = 48(x-1)^{-5}$$

$$y^n = (-1)^n 2(n!)(x-1)^{-(n+1)} = (-1)^n (n!) \frac{2}{(x-1)^{(n+1)}}$$

1) Ex. find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$

Sol.

To start , we differentiate both side of the eq. w.r.t. x to find $y' = \frac{dy}{dx}$

$$6x^2 - 6y\frac{dy}{dx} = 0$$

$\frac{dy}{dx} = \frac{x^2}{y}$ when $y \neq 0$

We now apply the Quotient Rule to find y''

2) $y'' = \frac{d^2y}{dx^2}$

$$= \frac{d}{dx}(y') = \frac{y(2x) - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2y'}{y^2}$$

Finally, we substitute $y' = \frac{x^2}{y}$ in the eq.

$y'' = \frac{2x}{y} - \frac{x^4}{y^3}$ when $y \neq 0$

Rule9:- power rule for fractional exponents

If U is a differentiable function of x and p and q are integers with $q > 0$ then

$$\frac{d}{dx}U^{\frac{p}{q}} = \frac{p}{q}U^{\frac{p}{q}-1}\frac{du}{dx} \quad \text{provided } U \neq 0$$

Ex.s

a) $y = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ when $x > 0$

b) $y = (3x^{\frac{1}{2}} + 5)^{-\frac{1}{4}}$

$$\frac{dy}{dx} = -\frac{1}{4}\left(3x^{\frac{1}{2}} + 5\right)^{-\frac{5}{4}}\left(\frac{3}{2}x^{-\frac{1}{2}}\right)$$

The Chain Rule (قاعدة السلسلة)

(Short form)

If y is a differentiable function of x and x is differentiable function of t , then y is differentiable function of t and
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Ex.

find $\frac{dy}{dt}$ at $t = -1$ if $y = x^3 + 5x - 4$ and $x = t^2 - 1$ Sol.

At $t = -1 \Longrightarrow x = 0$

$\left.\frac{dy}{dt}\right|_{t=-1} = \left.\frac{dy}{dx}\right|_{x=0} \cdot \left.\frac{dx}{dt}\right|_{t=-1} = (3x^2 + x).2t = 5(-2) = -10$

The Chain Rule

(First form)

Suppose that $h = g \circ f$ is the composite of differential function $y = g(x)$ and $x = f(t)$, then h is a differential function of t whose derivative at each value of t is

$$\underset{\text{at } t}{(g \circ f)'} = \underset{\text{at } x = f(t)}{g'} \quad . \quad \underset{\text{at } t}{f'}$$

In short, at each value of t

$$h'(t) = g'\big(f(t)\big).f'(t)$$

Ex. if $g(x) = \sqrt{x+2}$ and $x = f(t) = t^3 - 1$ find $\frac{d}{dx}(g \circ f)$ when $t = 2$

Sol.

$$\underset{\text{at } t=2}{(g \circ f)'} = \underset{\text{at } x=f(2)}{g'} \cdot \underset{\text{at } t=2}{f'}$$

$$\left.\frac{d}{dx}(g \circ f)\right|_{t=2} = g'(f(2)) \,.\, f'(2)$$

$$= \left.\frac{1}{2\sqrt{x+2}}\right|_{x=7} \cdot \left. 3t^2 \right|_{t=2}$$

$$= \frac{1}{6} \cdot 12 = 2$$

Ex.

If $y = t^2 - 1$ and $x = 2t + 3$ find $\frac{dy}{dx}$

Sol.

$\frac{dy}{dt} = 2t$, $\frac{dx}{dt} = 2 \Rightarrow \frac{dt}{dx} = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2t \cdot \frac{1}{2} = t$$

But $x = 2t + 3 \Rightarrow t = \frac{x-3}{2}$

Now we substitute t in the $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{x-3}{2}$$

Exercise:-

1) Find $\frac{dy}{dx}$

a) $y = \frac{(x^2+x)(x^2-x+1)}{x^4}$　　b) $y = (\frac{x+1}{x-1})^2$

b) c) $y = (x+1)^2(x^2+1)^{-3}$

2) $\frac{d^2y}{dx^2}$ if $y = (3-2x)^{-1}$

3) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by implicit defferentiation

a) $y^2 + 2y = 2x + 1$

b) $y + 2\sqrt{y} = x$

c) $y^2 + xy = 1$

4) Find $\frac{dy}{dt}$ and $\frac{d^2y}{dx^2}$ by Chain Rule. Expressing the results in terms of t

$y = x^4$, $x = \sqrt[3]{t}$

Applications of Derivatives

Rolle's Theorem مبرهنة رول

Let f be differentiable function on (a,b) and continuous on $[a,b]$, if $f(a) = f(b) = 0$ then there is at least one number c in (a,b) such that $f^{-}(c) = 0$.

Ex. $f(x) = x^2 - 2x - 3$ in $[-1,3]$

Sol. Clear that $f(x)$ is differentiable on (-1,3) and $f(x)$ is continuous on [-1,3] since (polynomial).

$$f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$$

$$f(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0$$

$$\therefore f(-1) = f(3) = 0$$

Apply Rolle's Theorem

$f'(x) = 2x - 2$

$2x - 2 = 0 \quad \Longrightarrow x = 1$

$\therefore f'(1) = 0$ and $-1 < 1 < 3$

Ex.

Show that $f(x) = \frac{x^2-x-6}{x-1}$ satisfies the hypothesis of the Rolle's Theorem on [-2, 3] and find all values of c in the interval (-2, 3).

Sol.

$f(x)$ is discontinuous at $x = 1$ and 1 is point of discontinuity since $\lim_{x\to 1} f(x) =$ not exist therefore we can't apply Rolle's Theorem.

Ex.

$f(x) = x^{\frac{2}{3}} - 2x^{\frac{1}{3}}$ in [0 , 8]

Sol.

$f'(x) = \frac{2}{3}x^{\frac{-1}{3}} - \frac{2}{3}x^{\frac{-2}{3}}$

HW.

Mean-Value Theorem مبرهنة القيمة الوسطى

Let f be differentiable function on (a, b) and continuous on $[a, b]$, then there is at least one number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Ex.

Show that $f(x) = \frac{x^3}{4} + 1$ satisfies the hypothesis of the Mean-Value Theorem on [0, 2] and find all values of *c* in the interval (0, 2).

Sol.

The function $f(x)$ is continuous and differentiable every where because it is a polynomial.

In particular, $f(x)$ is continuous on [0,2] and differentiable on (0,2), so the hypothesis is satisfied with $a = 0$, $b = 2$

$$f(a) = f(0) = 1 \quad , \quad f(ab) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4} \quad , \quad f'(c) = \frac{3c^2}{4}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{3c^2}{4} = \frac{3 - 1}{2 - 0} = 1 \quad \Longrightarrow 3c^2 = 4 \Longrightarrow c = \mp\frac{2}{\sqrt{3}}$$

Only the positive solution lies in the interval (0,2) therefor $c = \frac{2}{\sqrt{3}}$

Derivatives of Trigonometric Function

$\mathbf{\sin x}$ جيب الزاوية , $\mathbf{\cos x}$ جيب تمام الزاوية

$$\tan x = \frac{\sin x}{\cos x}$$ ظل الزاوية

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$ ظل تمام الزاوية

$\sec x = \frac{1}{\cos x}$ قاطع الزاوية , $\csc x = \frac{1}{\sin x}$ قاطع تمام الزاوية

$$\sin(-x) = -\sin x \quad , \quad \cos(-x) = \cos x$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin 2\theta = 2 \sin \theta \cos \theta \, , \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \, , \quad \csc^2\theta = 1 + \cot^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad , \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Ex.

Find $\frac{dy}{dx}$ if $y = x^2 \tan x$

Sol.

Using the product rule , we obtain

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\tan(x)) + \tan(x) \frac{d}{dx}(x^2)$$

$$= x^2 \sec^2 x + 2x \tan x$$

Ex.

Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}$

Sol.

Using the quotient rule , we obtain

$$\frac{dy}{dx} = \frac{(1+\cos x)\cdot\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

Ex.

Find $y''(\pi/4)$ if $y(x) = \sec x$ $\quad (\pi/4 = 45^\circ)$

Sol.

$y'(x) = \sec x \tan x$

$y''(x) = \sec x \sec^2 x + \tan x \cdot \sec x \tan x$

Thus

$$y''\left(\pi/4\right) = \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)$$

$$= (\sqrt{2})^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$$

Derivatives Logarithmic and Exponential Functions

Def.

The natural Logarithm of x is denoted by $\ln x$ and is defined by the integral.

$$\ln x = \int_1^x \frac{1}{t} dt \qquad , \quad x > 0$$

Theorem:-

For any positive numbers a and c and any Rational number :-

a) $\ln ac = \ln a + \ln c$
b) $\ln \frac{1}{c} = -\ln c$
c) $\ln \frac{a}{c} = \ln a - \ln c$
d) $\ln a^r = r \ln a$

> Rational no. is a number which can be expressed in the form $\frac{p}{q}$ wherein $q \neq 0$ and both p and q are integers.

Theorem:-

1- The domain of $\ln x$ is $(0, +\infty)$.
2- $\lim_{x \to 0^+} \ln x = +\infty$ and $\lim_{x \to +\infty} \ln x = +\infty$
3- The rang of $\ln x$ is $(-\infty, +\infty)$.

Derivatives Logarithmic:-

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad x > 0$$

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$ wherein u is a differentiable function of x

Ex.

Find $\frac{d}{dx}[\ln(x^2+1)]$

Sol.

$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) = \frac{1}{x^2+1} \cdot (2x) = \frac{2x}{x^2+1}$

Ex.

Find $\frac{d}{dx}\left[\ln(\frac{x^2 \sin x}{\sqrt{1+x}})\right]$

Sol.

$\frac{d}{dx}\left[\ln(\frac{x^2 \sin x}{\sqrt{1+x}})\right] = \frac{d}{dx}[\ln(x^2 \sin x) - \ln\sqrt{1+x}$

$= \frac{d}{dx}\left[\ln x^2 + \ln \sin x - \frac{1}{2}\ln(1+x)\right]$

$$= \frac{d}{dx}\left[2\ln x + \ln \sin x - \frac{1}{2}\ln(1+x)\right] = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$$

$= \frac{2}{x} + \cot x - \frac{1}{2(1+x)}$

Def.

The Inverse of the natural logarithm function $\mathbf{\ln x}$ is denoted by $\boldsymbol{e^x}$ and is called the natural exponential function.

Theorem:-

The natural exponential function $\boldsymbol{e^x}$ is differentiable on $(-\infty, +\infty)$ and it has derivative :-

$$\frac{d}{dx}[e^x] = e^x$$

Note:-

If u is a differentiable function of x ,then

$$\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

Ex. Find

1) $\frac{d}{dx}\left[e^{x^3}\right] = e^{x^3} \cdot \frac{d}{dx}(x^3) = 3x^2 e^{x^3}$
2) $\frac{d}{dx}\left[e^{\cos x}\right] = e^{\cos x} \cdot \frac{d}{dx}(\cos x) = -\sin x\, e^{\cos x}$

Theorem:-

1) $\lim_{x\to+\infty} e^x = +\infty$ 2) $\lim_{x\to+\infty} e^{-x} = 0$

3) $\lim_{x\to-\infty} e^x = 0$ 4) $\lim_{x\to-\infty} e^{-x} = +\infty$

Notes:-

$\ln 1 = 0$, $\ln e = 1$, $\ln\frac{1}{e} = -1$, $\ln e^2 = 2$

$\boldsymbol{\ln(e^x) = x}$ for all real values of x , $\boldsymbol{e^{\ln x} = x}$ for $x > 0$

Ex.

Solve the equation $e^{2x-6} = 4$ for x.

Sol.

we take the natural logarithm of both sides of the equation and use the rule $\boldsymbol{\ln(e^x) = x}$

$\ln(e^{2x-6}) = \ln 4$

$2x - 6 = \ln 4$

$2x = 6 + \ln 4$

$x = 3 + \frac{1}{2}\ln 4 = 3 + \ln 4^{\frac{1}{2}}$

$\ln a^r = r \ln a$

$x = 3 + \ln 2$

Theorem:-

L'Hôpital's rule for form $\frac{0}{0}$ (قاعدة لوبيتال)

Suppose that f and g are differentiable function on an open interval containing $x = a$, except possibly at $x = a$ and that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ has a finite limit or if this limit is $-\infty$ or $+\infty$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$ or as $x \to +\infty$.

Ex.

$\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$

$= \lim_{x\to 0} \frac{\sin x}{1+2x}$ by L'Hôpital's rule

$= \frac{0}{1} = 0$

Ex.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2} \qquad (\text{still } \frac{0}{0})$$

$$=\lim_{x\to 0}\frac{\sin x}{6x} \qquad (\text{still } \frac{0}{0})$$

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$

Theorem:-

L'Hôpital's rule for form $\frac{\infty}{\infty}$ (قاعدة لوبيتال)

Suppose that f and g are differentiable function on an open interval containing $x = a$, except possibly at $x = a$ and that

$$\lim_{x\to a} f(x) = \infty \quad \text{and} \quad \lim_{x\to a} g(x) = \infty$$

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ has a finite limit or if this limit is $-\infty$ or $+\infty$ then

$$\mathbf{\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$ or as $x \to +\infty$.

Exercises:

1) $\lim_{x\to 2}\frac{x^2-4}{x-2}$
2) $\lim_{x\to 0}\frac{\sin 2x}{x}$
3) $\lim_{x\to 0}(\frac{1}{x}-\frac{1}{\sqrt{x}})$

4) $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1}$
5) $\frac{d}{dx}[\ln|\sin x|]$
6) $\frac{d}{dx}[x^3 e^x]$
7) $\frac{d}{dx}[\frac{e^x}{\ln x}]$
8) $\frac{d}{dx}[\sin^2(\ln x)]$

Integration (التكامل)

A special symbol is used to denote the collection of all antiderivatives of a function f.

Def. The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)dx$$

The symbol 1 is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Note.
1) $\int dx = x + c$
2) $\int a\, dx = a \int dx$
3) $\int (dx + dy) = \int dx + \int dy$
4) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad , n \neq -1$
5) $\int \frac{1}{x} dx = \ln|x| + c = \ln x + c \qquad , x > 0$
6) $\int e^x dx = e^x + c$
7) $\int a^x dx = \frac{a^x}{\ln a} + c$ **note:** $\frac{d}{dx}(a^x) = a^x \ln a$

Ex.

$$\int 2x dx = x^2 + c$$

$$\int (x^2 - 2x + 5)dx = \frac{x^3}{3} - x^2 + 5x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int (sec^2 x + \frac{1}{2\sqrt{x}})dx = \tan x + \sqrt{x} + c$$

Integration by parts (تكامل بالتجزئة)

The integration by parts formula

$$\int \boldsymbol{u(x)v'(x)dx} = \boldsymbol{u(x)v(x)} - \int \boldsymbol{v(x)u'(x)dx}$$

Ex.Find $\int x \cos x \, dx$

Sol.

There is no obvious antiderivative of $x \cos x$, so we use the integration by parts formula.

$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. In this case we factor the expression $x \cos x$ into

$u(x) = x$ and $v'(x) = \cos x$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$u'(x) = 1$ and $v(x) = \sin x$

$$\int \underset{u(x)}{x} \underset{v'(x)}{\cos x} \, dx = \underset{u(x)}{x} \underset{v(x)}{\sin x} - \int \underset{v(x)}{\sin x} \underset{u'(x)}{(1)} \, dx$$

$= x \sin x + \cos x + c$

There are four apparent choices available for $u(x)$ and $v'(x)$ in Example.

1. Let $u(x) = 1$ and $v'(x) = x\cos x$.
2. Let $u(x) = x$ and $v'(x) = \cos x$
3. Let $u(x) = x\cos x$ and $v'(x) = 1$
4. Let $u(x) = \cos x$ and $v'(x) = x$.

Choice 2 was used in Ex. The other three choices lead to integrals we don’t know how to integrate. For instance, Choice 3, with $u'(x) = \cos x - x\sin x$, leads to the integral.

$$\int (x\cos x - x^2 \sin x)dx$$

ملحوظة:

الهدف من التكامل بالتجزئة هو الأنتقال من الصيغة المعطاة في السؤال التي لا نمتلك رؤية واضحة لحلها إلى صيغة أبسط وأوضح . وبصورة عامة، نختار أولاً $v'(x)$ بحيث نستطيع أن نجري عليها التكامل بسهولة والجزء المتبقي نختاره $u(x)$ بحيث نستطيع ايجاد $v(x)$ من $v'(x)$.

Ex.

Find $\int \ln x\, dx$

Sol.

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Let $u(x) = \ln x$ and $v'(x) = 1$

$$\int \underbrace{\ln x}_{u(x)} \cdot \underbrace{1}_{v'(x)}\, dx = \underbrace{(\ln x)}_{u(x)}\underbrace{x}_{v(x)} - \int \underbrace{x}_{v(x)} \underbrace{\frac{1}{x}}_{u'(x)}\, dx$$

$$= x\ln x - x + c$$

The formula is often given in differential form. With $v'(x)dx = dv$ and $u'(x)dx = du$, the integration by parts formula becomes

$$\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{vdu}$$

Ex.

Find $\int x^2 e^x dx$

Sol.

Let $u(x) = x^2$ and $v'(x) = e^x$

$\therefore \quad u'(x) = 2x$ and $v(x) = e^x$

Or if we use second formula $\int u\,dv = uv - \int vdu$

$u = x^2 \;\Longrightarrow du = 2xdx$

$v = e^x \;\Longrightarrow dv = e^x dx$

$$\int \underset{u}{x^2}\,\underbrace{e^x\,dx}_{dv} = \underset{u}{x^2}\,\underset{v}{e^x} - \int \underset{v}{e^x}\,\underbrace{2x\,dx}_{du}.$$

Integration by parts formula

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int \underset{u}{x}\,\underbrace{e^x\,dx}_{dv} = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x}\,\underbrace{dx}_{du} = xe^x - e^x + C.$$

Integration by parts Equation (2)
$u = x,\ dv = e^x\,dx$
$v = e^x, \quad du = dx$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2xe^x + 2e^x + C,\end{aligned}$$

Ex.Find $\int x^2 e^x dx$

Sol.Let $u(x) = x^2$ and $v'(x) = e^x$

$\therefore \quad u'(x) = 2x$ and $v(x) = e^x$

Or if we use second formula $\int u\, dv = uv - \int v du$

$u = x^2 \implies du = 2xdx$

$v = e^x \implies dv = e^x dx$

$$\int \underbrace{x^2}_{u} \underbrace{e^x\, dx}_{dv} = \underbrace{x^2}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{2x\, dx}_{du}.$$

Integration by parts formula

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\, dx$. Then $du = dx$, $v = e^x$, and

$$\int \underbrace{x}_{u} \underbrace{e^x\, dx}_{dv} = \underbrace{x}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du} = xe^x - e^x + C.$$

Integration by parts Equation (2)
$u = x,\ dv = e^x\, dx$
$v = e^x, \quad du = dx$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x\, dx &= x^2 e^x - 2\int xe^x\, dx \\ &= x^2 e^x - 2xe^x + 2e^x + C,\end{aligned}$$

Area Bounded by Two Curves

(المساحة المحددة بين منحنيين)

Total area $=\int_a^b [f_1(x) - f_2(x)]dx$

Ex. Find the area a bounded by the parabola $y = 2 - x^2$
, and the straight line $y = -x$.

Sol. Points of intersections are given by :-
$y = 2 - x^2 = -x$
$x^2 - x - 2 = 0$
$(x + 1) (x - 2) = 0$
$x = -1$ or $x = 2$
So the area a bounded by the two curves is given by :-

Total area $=\int_a^b [f_1(x) - f_2(x)]dx$
$= \int_{-1}^{2} (2 - x^2 + x) \, dx$
$= [2x - \frac{x^3}{3} + \frac{x^2}{2}]_{-1}^{2} = 4.5$

==

(Integration by Partial Fraction)
(التكامل بواسطة تجزئة الكسور)

في العادة نعبر عن المجموع الجبري لاي عدد من الكسور بدلالة كسر واحد وهذه العملية تسمى اختصار الكسور مثال :-

$$\frac{13}{x-2} - \frac{10}{x-1} = \frac{13\,(x-1) - 10\,(x-2)}{(x-2)\,(x-1)}$$

ونبحث الان العملية العكسية وهي كيفية تحليل كسر معلوم الى كسور ابسط منه تعرف بكسوره الجزئية ولتحليل اي كسر الى كسوره الجزئية بحيث يكون كسرا مناسبا بمعنى ان درجة البسط اقل من درجة المقام ويحلل الكسر المناسب الى كسوره الجزئية كما يلي :-

1- نحلل المقام اي يكتب على شكل حاصل ضرب اقواس .
2- كل قوس من الدرجة الاولى في المقام يقابله كسر واحد مقامه القوس المذكور وبسطه مقدار ثابت
3- كل قوس من الدرجة الاولى مكرر مرتان يقابله كسران الاول مقامه القوس والثاني مقامه مربع القوس وبسط كلا منهما مقدار ثابت وهكذا اذا كان القوس مكرر (n) من المرات يقابله (n) من

الكسور مقاماتها على الترتيب :- القوس ، القوس تربيع ، القوس تكعيب ،.............، القوس اس n وبسط كلا منها مقدار ثابت .

Ex. $$\frac{1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(X+1)^2} + \frac{C}{(X+1)^3}$$

حيث : A , B , C ثوابت

4-نكتب الكسر الاصلي كمجموع الكسور الجزئية السابقة ذات العوامل المجهولة في بسط كلا منها .

5-نضرب الطرفين في مقام الكسر الاصلي فنحصل على معادلة بين متعددات حدود بمقارنة معاملات القوى المتناظرة في الطرفين نحصل على العوامل المجهولة .

Ex. $$\frac{1}{x^2-x-6} = \frac{1}{(x-3)(x+2)} = \frac{A}{X-3} + \frac{B}{X+2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$= \frac{AX+2A+BX-3B}{(x-3)(x+2)} = \frac{(A+B)X+2A-3B}{(x-3)(x+2)}$$

الان نقارن بين بسطي الطرفين :-

$$1 = (A+B)X + 2A - 3B$$

بما انه لا يوجد X في الطرف الايسر فهذا يعني ان :-

$$A + B = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

والحد الثابت 1 يقابله 2A – 3B اي ان : -

$$2A - 3B = 1 \quad \ldots\ldots\ldots\ldots (2)$$

نحل (1) و (2) بالتعويض او بالحذف :-

$$A + B = 0$$

$$A = -B$$

نعوض في المعادلة (2) فنحصل على :

$$2(-B) - 3B = 1$$

$$-5B = 1$$

$$B = -\frac{1}{5}$$

نعوض قيمة B في (1) فنحصل على : $A = \frac{1}{5}$

اي ان :- $$\frac{1}{x^2-x-6} = \frac{\frac{1}{5}}{X-3} - \frac{\frac{1}{5}}{X+2}$$

الان نستخدم هذه الطريقة في ايجاد قيمة التكامل عندما نحتاج الى تجزئة الكسور وكما موضح في المثال الاتي :-

Ex. $\int \frac{x\,dx}{(x-1)(x-2)}$

$$\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$
$$= \frac{AX-2A+BX-B}{(x-1)(x-2)} = \frac{(A+B)X-2A-B}{(x-1)(x-2)}$$

بمقارنة البسطين :

X = (A + B) X – 2A – B
A + B = 1
A = 1 – B
– 2 A – B = 0 ……………(1)

نعوض قيمة A في المعادلة (1) فنحصل على :-

–2(1 – B) – B = 0
B = 2

نعوض عن قيمة B فنحصل على قيمة A : A = 1– 2 = – 1

نعوض عن قيمة A , B فنحصل على :-

$$\int \frac{x\,dx}{(x-1)(x-2)} = \int \left(\frac{-1}{(x-1)} + \frac{2}{X-2}\right) dx = -Ln\,(x-1) + 2\,Ln\,(x-2) + c$$

Ex . $\int \frac{dx}{x^2+2x-8} = \int \frac{dx}{(x-2)(x+4)}$

$$\frac{1}{(x-2)(x+4)} = \frac{A}{X-2} + \frac{B}{x+4} = \frac{A(X+4)+B(X-2)}{(x-2)(x+4)} = \frac{AX+4A+BX-2B}{(x-2)(x+4)}$$
$$= \frac{(A+B)X+4A-2B}{(x-2)(x+4)}$$

بمقارنة البسطين :-

1 = (A + B) X +4A –2 B
A + B = 0
A = – B
4A – 2B = 1 ……………(1)

نعوض قيمة A في المعادلة (1) فنحصل على :-

– 4B – 2B = 1
B = $\frac{-1}{6}$

نعوض عن قيمة B فنحصل على قيمة A : A = $\frac{1}{6}$

نعوض عن قيمة A , B فنحصل على :-

$$\int \frac{dx}{(x-2)(x+4)} = \int \left(\frac{\frac{1}{6}}{X-2} + \frac{\frac{-1}{6}}{X+4}\right) dx = \frac{1}{6} Ln\,(x-2) - \frac{1}{6} Ln\,(x+4) + c$$